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载荷垂直于非均匀变截面弹性园环的弯扭问题

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摘　要

本文从基本微分方程出发，利用了内力、位移的微分关系，全面研究载荷垂于非均匀变截面弹性园环的弯扭问题，通过适当的数学变换；获得了抗弯刚度和抗扭刚度为任意函数的通用积分式。并对两个特殊情况——阶梯园环和等截面园环获得了通解。为了说明该公式的具体应用，文末给出了示例，并作了求解。

一、引　言

在化工设备、大型管道及各种框架中，在飞机，潜艇及各种机器上，园环是一种重要的结构元件。为了对这些元件进行正确而又合理的设计，必须计算在各种载荷和支承条件下的应力及挠度。园环的弯曲〔1〕〔2〕〔3〕等曾作过很多的研究。由于非均匀变截面问题常常导出变系数微分方程。它们的求解有时遇到很大的数学困难。首先，本文由微元体平衡得到的内力微分关系式，施行*Laplace*变换和逆变换，保证了内力的连续。其次，把内力与位移的微分关系联系起来，再一次施行*Laplace*变换和逆变换。保证了位移的连续，从而避免了求解变系数微分方程，获得了该弯扭问题的通用积分式。对其特殊情况——阶梯园环及等截面园环通过积分得到了一般解。

本文公式它建立在下列规定的基础上：

（一）园环的半径远大于它的径向厚度，因而可以应用直梁的挠曲理论。

（二）任何处的应力极限均低于弹性极限。

（三）变形后仍不失为园环，即小变形或小挠度理论。

此外，本文不计轴力以及园环沿功向的变形。

二、非均匀变截面弹性园环的弯扭通用积分式

（一）园环的内力及其微分关系

在图1所示的极坐标系中，园环任一截面作用的内力有弯矩$M(\varphi)$、扭矩$T(\varphi)$，剪力$P(\varphi)$，均布载荷q。由微元体的平衡得到。

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$$\left.\begin{aligned}\frac{dT(\varphi)}{d\varphi}&=M(\varphi)\\ \frac{dM(\varphi)}{d\varphi}&=-T(\varphi)+RP(\varphi)\\ \frac{dP(\varphi)}{d\varphi}&=q\end{aligned}\right\}\qquad(2\text{—}1)$$

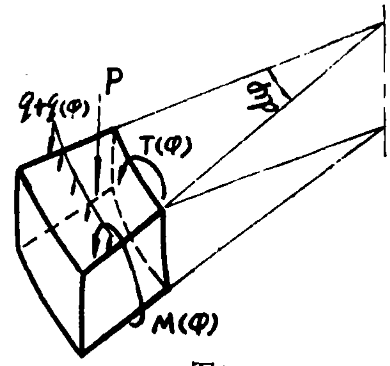

图1

解（2—1）式得：

$$\frac{d}{d\varphi^2}\left\{\frac{d^2}{d\varphi^2}+1\right\}T(\varphi)=R^2q\qquad(2\text{—}2)$$

初始边界条件

$$\left.\begin{aligned}T(\varphi)\,|_{\Phi=0}&=-T_0\\ M(\varphi)\,|_{\Phi=0}&=M_0\\ \left\{\frac{d^2}{d\varphi^2}+1\right\}T(\varphi)_{\Phi=0}&=RP_0\end{aligned}\right\}\qquad(2\text{—}3)$$

对（2—3）式施行*Laplace*变换，并用卷积定理，再径逆变换后，得其内力解。

$$\left.\begin{aligned}T(\varphi)&=RP_0(1-\cos\varphi)-T_0\cos\varphi+M_0\sin\varphi+qR^2(\varphi-\sin\varphi)+T^*(\varphi)\\ M(\varphi)&=RP_0\sin\varphi+T_0\sin\varphi+M_0\cos\varphi+qR^2(1-\cos\varphi)+M^*(\varphi)\end{aligned}\right\}\qquad(2\text{—}4)$$

如图2所示，引入*Heviside***函数$H(\varphi-\varphi_j)$（$j=a$、b、c、d）则$T^*(\varphi)$，$M^*(\varphi)$分别是

$$\begin{aligned}T^*(\varphi)=&H(\varphi-\varphi_a)RP_1[1-\cos(\varphi-\varphi_a)]-H(\varphi-\varphi_b)T_1\cos(\varphi-\varphi_b)+\\ &+H(\varphi-\varphi_c)M_1\sin(\varphi-\varphi_c)+H(\varphi-\varphi_d)q^*R^2[\varphi-\varphi_d-\sin(\varphi-\varphi_d)]\end{aligned}\qquad(2\text{—}5)$$

$$\begin{aligned}M^*(\varphi)=&H(\varphi-\varphi_a)RP_1\sin(\varphi-\varphi_a)+H(\varphi-\varphi_b)T_1\sin(\varphi-\varphi_b)+\\ &+M(\varphi-\varphi_c)M_1\cos(\varphi-\varphi_c)+H(\varphi-\varphi_{d.})R^2q[1-\cos(\varphi-\varphi_{d.})]\end{aligned}\qquad 2(\text{—}6)$$

式中$M(\varphi)$——在角φ处，园环的弯矩。

$T(\varphi)$——在角φ处，园环的扭矩。

P　——垂直于园环的集中载荷。

q——垂直于园环的均布载荷。

To、Mo、Po均为内力的初参数

（二）园环的位移及其微分关系

载荷垂直于曲率平面的园环，其变形特征通常是设计中的一个重要问题，通常该问题的计算用单位载荷法或Castiano定理。但这种方法只局限于环上的一点。若设计需要了解一系列的点，这就需要进行一系列的繁锁计算。尤其在变截面中，其计算更为复杂。本文用弯

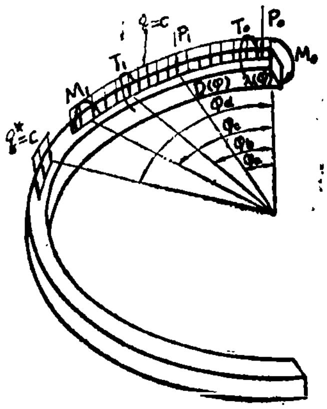

图2

注：***Hevisde*函数　$H(\varphi-\varphi_j)\begin{cases}=1 & \varphi>\varphi_j\\ =0 & \varphi\leqslant\varphi_j\end{cases}$

矩和扭矩描述的微分方程就可避免上述的缺陷。

园环中性轴上任一点及与该点相联系的微元体，变形后它们的位置由以下三个位移分量确定[4]。

$$\left.\begin{aligned}
&\frac{M(\varphi)}{D(\varphi)}=\frac{\beta(\varphi)}{R}-\frac{1}{R^2}\frac{d^2U_z(\varphi)}{d\varphi^2}\\
&\frac{T(\varphi)}{\lambda(\varphi)}=\frac{1}{R}\frac{d\beta(\varphi)}{d\varphi}+\frac{1}{R^2}\frac{dU_z(\varphi)}{d\varphi^2}\\
&\alpha(\varphi)=\frac{1}{R}\frac{dU_z(\varphi)}{d\varphi}
\end{aligned}\right\} \tag{2—7}$$

式中：$U_z(\varphi)$ ——园环的位移

$\alpha(\varphi)$ ——园环的倾角

$\beta(\varphi)$ ——园环的扭转角

$D(\varphi)=E(\varphi)\cdot I(\varphi)$ ——抗弯刚度

$\lambda(\varphi)=K(\varphi)\cdot G(\varphi)$ ——抗扭刚度

E——弹性模量　　I——惯性矩

K——形状因子　　G——剪切模量

解（2—7）式得到一般挠曲方程：

$$\frac{d}{d\varphi}\left\{\frac{d^2}{d\varphi^2}+1\right\}U_z(\varphi)=-R^2\left\{\frac{d}{d\varphi}\left(\frac{M(\varphi)}{D(\varphi)}\right)-\frac{T(\varphi)}{\lambda(\varphi)}\right\} \tag{2—8}$$

式（2—8）的初始边界条件是

$$\left.\begin{aligned}
&U_z(\varphi)\big|_{\varphi=0}=U_{zo}\\
&U_z'(\varphi)\big|_{\varphi=0}=R\alpha_o\\
&U_z''(\varphi)\big|_{\varphi=0}=R\beta_o
\end{aligned}\right| \tag{2—9}$$

对（2—8）式施行$LapLace$变换，并用卷积定理，再经逆变换后，得其一般解。

$$\left.\begin{aligned}
Uz(\varphi)=&Uzo+R\alpha_o\sin\varphi+R\beta_o(1-\cos\varphi)-R^2\int_0^\varphi\frac{M(\zeta)}{D(\zeta)}\\
&\cdot\sin(\varphi-\zeta)d\zeta+R^2\int_0^\varphi\frac{T(\zeta)}{\lambda(\zeta)}[1-\cos(\varphi-\zeta)]d\zeta\\
R\alpha(\varphi)=&R\alpha_o\cos\varphi+R\beta_o\sin\varphi-R^2\int_0^\varphi\frac{M(\zeta)}{D(\zeta)}\cos(\varphi-\zeta)d\zeta+\\
&+R^2\int_0^\varphi\frac{T(\zeta)}{\lambda(\zeta)}\sin(\varphi-\zeta)d\zeta\\
R\beta(\varphi)=&R\beta_o\cos\varphi-R\alpha_o\sin\varphi+R^2\int_0^\varphi\frac{M(\zeta)}{D(\zeta)}\sin(\varphi-\zeta)d\zeta+\\
&+R^2\int_0^\varphi\frac{T(\zeta)}{\lambda(\zeta)}\cos(\varphi-\zeta)d\zeta
\end{aligned}\right| \tag{2—10}$$

用（2—4），（2—5）、（2—6）式代去（2—10）式，就可得其如图3所示的载荷垂直于非均匀变截面弹性圆环弯扭的通用积分式。

$$U_z(\varphi)=U_{z0}+R\alpha\sin\varphi+R\beta_0(1-\cos\varphi)+$$

$$+R^3\int_0^{\varphi}\frac{P_0}{2D(\zeta)}\left[\cos\varphi-\cos(2\zeta-\varphi)\right]d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{T_0}{2D(\zeta)}\left[\cos\varphi-\cos(2\zeta-\varphi)\right]d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{M_0}{2D(\zeta)}\left[\sin(2\zeta-\varphi)-\sin\varphi\right]d\zeta+$$

$$+R^4\int_0^{\varphi}\frac{q}{D(\zeta)}\left\{-\sin(\varphi-\zeta)-\right.$$

$$\left.-\frac{1}{2}\left[\sin(2\zeta-\varphi)-\sin\varphi\right]\right\}d\zeta+$$

$$+R^3\int_0^{\varphi}\frac{P_0}{\lambda(\zeta)}\left\{1-\cos\zeta-\cos(\varphi-\zeta)+\right.$$

$$\left.+\frac{1}{2}\left[\cos\varphi+\cos(2\zeta-\varphi)\right]\right\}d\zeta+$$

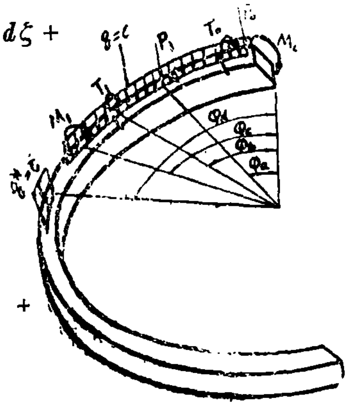

图3

$$+R^2\int_0^{\varphi}\frac{T_0}{\lambda(\zeta)}\left\{-\cos\zeta+\frac{1}{2}\left[\cos\varphi+\cos(2\zeta-\varphi)\right]\right\}d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{M_0}{\lambda(\zeta)}\left\{\sin\zeta-\frac{1}{2}\left[\sin(2\zeta-\varphi)+\sin\varphi\right]\right\}d\zeta+$$

$$+R^4\int_0^{\varphi}\frac{q^*}{\lambda(\zeta)}\left\{\zeta-\sin\zeta-\zeta\cos(\varphi-\zeta)+\frac{1}{2}\left[\sin(2\zeta-\varphi)+\right.\right.$$

$$\left.\left.+\sin\varphi\right]\right\}d\zeta+$$

$$+R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{2D(\zeta)}\left[\cos(\varphi-\varphi_a)-\cos(2\zeta-\varphi-\varphi_a)\right]d\zeta+$$

$$+R^2\int_{\varphi_b}^{\varphi}\frac{T_1}{2D(\zeta)}\left[\cos(\varphi-\varphi_b)-\cos(2\zeta-\varphi-\varphi_b)\right]d\zeta+$$

$$+R^2\int_{\varphi_c}^{\varphi}\frac{M_1}{2D(\xi)}\left[\sin(2\xi-\varphi-\varphi_c)-\sin(\varphi-\varphi_c)\right]d\xi+$$

$$+R^4\int_{\varphi_d}^{\varphi}\frac{q}{D(\xi)}\left\{-\sin(\varphi-\zeta)-\frac{1}{2}\sin(2\xi-\varphi-\varphi_d)-\right.$$

$$\left.\sin(\varphi-\varphi_d)\right]\Big\}d\xi+R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{\lambda(\xi)}\left\{1-\cos(\xi-\varphi_a)-\cos\right.$$

$$\left.(\varphi-\xi)+\frac{1}{2}\left[\cos(\varphi-\varphi_a)+\cos(2\xi-\varphi-\varphi_a)\right]\right\}d\xi+$$

$$+R^2\int_{\varphi_b}^{\varphi}\frac{T_1}{\lambda(\xi)}\{-\cos(\xi-\varphi_b)+\frac{1}{2}[\cos(2\xi-\varphi-\varphi_b)+\cos(\varphi-\varphi_b)]\}d\xi+R^2\int_{\varphi_c}^{\varphi}\frac{M_1}{\lambda(\xi)}\{\sin(\xi-\varphi_c)-\frac{1}{2}[\sin(2\xi-\varphi-\varphi_c)+\sin(\varphi-\varphi_c)]\}d\xi+R^4\int_{\varphi_d}^{\varphi}\frac{q^*}{\lambda(\xi)}\{(\xi-\varphi_d)-(\xi-\varphi_d)\cos(\varphi-\xi)-\sin(\varphi-\varphi_d)+\frac{1}{2}[\sin(2\xi-\varphi-\varphi_d)+\sin(\varphi-\varphi_d)]\}d\xi \tag{2-11}$$

$$R\alpha(\varphi)=R\alpha_0\cos\varphi+R\beta_0\sin\varphi-R^3\int_0^{\varphi}\frac{P_0}{2D(\zeta)}[\sin(2\zeta-\varphi)+\sin\varphi]d\zeta-$$

$$-R^2\int_0^{\varphi}\frac{T_0}{2D(\zeta)}[\sin(2\zeta-\varphi)+\sin\varphi]d\zeta-$$

$$-R^2\int_0^{\varphi}\frac{M_0}{2D(\zeta)}[\cos\varphi+\cos(2\zeta-\varphi)]d\zeta-$$

$$-R^4\int_0^{\varphi}\frac{q}{D(\zeta)}\{\cos(\varphi-\zeta)-\frac{1}{2}[\cos\varphi+\cos(2\zeta-\varphi)]\}d\zeta+$$

$$+R^3\int_0^{\varphi}\frac{P_0}{\lambda(\zeta)}\{\sin(\varphi-\zeta)+\frac{1}{2}[\sin(2\zeta-\varphi)-\sin\varphi]\}d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{T_0}{2\lambda(\zeta)}[\sin(2\zeta-\varphi)-\sin\varphi]d\varphi+$$

$$+R^2\int_0^{\varphi}\frac{M_0}{2\lambda(\zeta)}\{-[\cos\varphi-\cos(2\zeta-\varphi)]\}d\zeta+$$

$$+R^4\int_0^{\varphi}\frac{q}{\lambda(\zeta)}\{\zeta\sin(\varphi-\zeta)+\frac{1}{2}[\cos\varphi-\cos(2\zeta-\varphi)]\}d\zeta-$$

$$-R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{2D(\zeta)}[\sin(2\zeta-\varphi-\varphi_a)+\sin(\varphi-\varphi_a)]d\zeta-$$

$$-\frac{R^2}{2}\int_{\varphi_b}^{\varphi}\frac{T_1}{D(\zeta)}[\sin(2\zeta-\varphi-\varphi_b)+\sin(\varphi-\varphi_b)]d\zeta-$$

$$-\frac{R^2}{2}\int_{\varphi_c}^{\varphi}\frac{M_1}{D(\zeta)}[\cos(\varphi-\varphi_c)+\cos(2\zeta-\varphi-\varphi_c)]d\zeta-$$

$$-\frac{R^4}{2}\int_{\varphi_d}^{\varphi}\frac{q^*}{D(\zeta)}\{\cos(\varphi-\zeta)-\frac{1}{2}[\cos(\varphi-\varphi_d)+\cos(2\zeta-\varphi-\varphi_d)]\}d\zeta+R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{\lambda(\zeta)}\{\sin(\varphi-\zeta)+\frac{1}{2}[\sin(2\zeta-\varphi-\varphi_a)-\sin(\varphi-\varphi_a)]\}d\zeta+\frac{R^2}{2}\int_{\varphi_b}^{\varphi}\frac{T_1}{\lambda(\zeta)}[\sin(2\zeta-\varphi-\varphi_b)-$$

$$-\sin(\varphi-\varphi_b)]d\zeta+\frac{R^2}{2}\int_{\varphi_c}^{\varphi}\frac{M_1}{\lambda(\zeta)}\{-[\cos(\varphi-\varphi_c)-$$

$$-\cos(2\xi-\varphi-\varphi_c)]\}d\zeta+R^4\int_{\varphi_d}^{\varphi}\frac{q^*}{\lambda(\xi)}\Big\{(\zeta-\varphi_d)\cdot\sin(\varphi-\zeta)+$$

$$+\frac{1}{2}[\cos(\varphi-\varphi_d)-\cos(2\zeta-\varphi-\varphi d)]\}d\zeta \qquad (2\text{—}12)$$

$$R\beta(\varphi)=-R\alpha_0\sin\varphi+R\beta_0\cos\varphi-R^3\int_0^{\varphi}\frac{P}{2D(\zeta)}[\cos\varphi-\cos(2\zeta-\varphi)]d\zeta-$$

$$-R^2\int_0^{\varphi}\frac{T_0}{2D(\zeta)}[\cos\varphi-\cos(2\zeta-\varphi)]d\zeta-$$

$$-R^2\int_0^{\varphi}\frac{M_0}{2D(\zeta)}[\sin(2\zeta-\varphi)-\sin\varphi]d\zeta+$$

$$+R^4\int_0^{\varphi}\frac{q}{D(\zeta)}\{+\sin(\varphi-\zeta)+\frac{1}{2}[\sin(2\zeta-\varphi)-\sin\varphi]\}d\zeta+$$

$$+R^3\int_0^{\varphi}\frac{P_0}{2\lambda(\zeta)}[2\cos(\varphi-\zeta)-\cos\varphi-\cos(2\zeta-\varphi)]d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{T_0}{2\lambda(\zeta)}[\cos\varphi+\cos(2\zeta-\varphi)]d\zeta+$$

$$+R^2\int_0^{\varphi}\frac{M_0}{2\lambda(\zeta)}[\sin(2\zeta-\varphi)+\sin\varphi]d\zeta+$$

$$+R^4\int_0^{\varphi}\frac{q}{\lambda(\zeta)}\{\zeta\cos(\varphi-\zeta)-\frac{1}{2}[\sin(2\zeta-\varphi)+\sin\varphi]\}d\zeta-$$

$$-R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{2D(\zeta)}[\cos(\varphi-\varphi_a)-\cos(2\zeta-\varphi-\varphi_a)]d\zeta-$$

$$-R^2\int_{\varphi_b}^{\varphi}\frac{T_1}{2D(\zeta)}[\cos(\varphi-\varphi_b)-\cos(2\zeta-\varphi-\varphi_b)]d\zeta-$$

$$-R^2\int_{\varphi_c}^{\varphi}\frac{M_1}{2D(\zeta)}[\sin(2\zeta-\varphi-\varphi_c)-\sin(\varphi-\zeta)]d\zeta+$$

$$+R^4\int_{\varphi_d}^{\varphi}\frac{q^*}{D(\zeta)}\{\sin(\varphi-\zeta)+\frac{1}{2}[\sin(2\zeta-\varphi-\varphi_d)-$$

$$-\sin(\varphi-\varphi_d)]\}d\zeta+R^3\int_{\varphi_a}^{\varphi}\frac{P_1}{2\lambda(\zeta)}[2\cos(\varphi-\zeta)-\cos(\varphi-\varphi_a)-$$

$$-\cos(2\zeta-\varphi-\varphi_a)]d\zeta+R^2\int_{\varphi_b}^{\varphi}\frac{T_1}{2\lambda(\zeta)}$$

$$[\cos(\varphi-\varphi_b)+\cos(2\zeta-\varphi-\varphi_b)]d\zeta+R^2\int_{\varphi_c}^{\varphi}\frac{M_1}{2\lambda(\zeta)}[\sin(2\zeta-\varphi-$$

$$\varphi_c)+\sin(\varphi-\varphi_c)]d\zeta-R^4\int_{\varphi_d}^{\varphi}\frac{q^*}{\lambda(\zeta)}\{(\zeta-\varphi)\cos(\varphi-\zeta)-$$

$$-\frac{1}{2}[\sin(2\zeta-\varphi-\varphi_d)+\sin(\varphi-\varphi_d)]\}d\zeta \quad (2\text{—}13)$$

式中φ_j——是初始点分别到载荷作用处的角度，（$j=a$、b、c、d）

U_{zo}、α_o、β_o——均为位移的初参数。

几点说明：1、当q、q^*间断分布时，而所求截面越过间断分布时，如图4所示，欲求φ_k处截面的位移。则必须作反向补偿。图中的虚线即是反向补偿。注意本文没有计入。

2、当载荷几重出现时，按几重形式叠加，如有5个集中载荷引起的位移。以求$U_z(\varphi)$为例，其表达式可描述为

$$R^3\sum_{k=1}^{5}P_k\left\{\int_{\varphi_{ak}}^{\varphi}\frac{1}{2D(\zeta)}[\cos(\varphi-\varphi_{ak})-\cos(2\zeta-\varphi-\varphi_{ak})]d\zeta+\right.$$

$$+\int_{\varphi_{ak}}^{\varphi}\frac{1}{\lambda(\zeta)}\{1-\cos(\zeta-\varphi_{ak})-\cos(\varphi-\zeta)+\frac{1}{2}[\cos(\varphi-$$

$$\left.-\varphi_{ak})+\cos(2\zeta-\varphi-\varphi_{ak})]\}d\zeta\right\}。$$

3、当抗弯，抗扭刚度是以指数形式描述时。则本文公式可用初等函数表达之。

4、当抗弯、抗扭刚度分别是以$D_o(1+\Omega_o\frac{\varphi}{2\pi})^m$、$\lambda_o(1+\Omega_1\frac{\varphi}{2\pi})^s$描述时。$\Omega_o$、$\Omega_1$——常系数。并且当$s$、$m=-1$、$-2$、……时，则本文公式可用初等函数表达之，当$s$、$m=1$、$2$……时，则本文公式可用正弦和余弦积分描述的特殊函数表达之。因篇幅关系，具体求解，这儿从略。

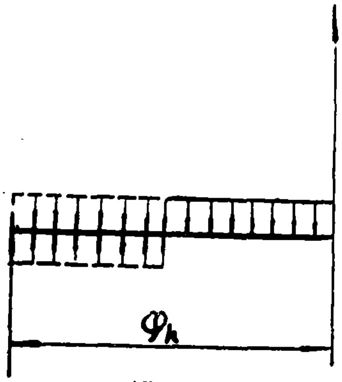

图4

在工程中，弹性园环的抗弯、抗扭刚度有时无法用指数及$D_o(1+\Omega_o\frac{\varphi}{2\pi})^m$、$\lambda_o(1+\Omega_1\frac{\varphi}{2\pi})^s$描述时，则可用阶梯函数作近似的逼近。以下我们给出阶梯园环的通解公式。

三、阶梯园环弯扭的通解

如图5所示经分段积分，并整理后得其阶梯园环弯扭的一般解。

$$U_z(\varphi)=U_{zo}+R\alpha_o\sin\varphi+R\beta_o(1-\cos\varphi)+\frac{R^3P_o}{4}\left\{-\sin\varphi\left(\frac{1}{D_1}+\right.\right.$$

$$\left.+\frac{1}{D_N}\right)+2\cos\varphi\cdot\varphi/D_n+\sum_{i=1}^{n-1}[2\cos\varphi\cdot\varphi_i-\sin(2\varphi_i-\varphi)]\left(\frac{1}{D_i}-\right.$$

$$\left.\frac{1}{D_{i+1}}\right)+\frac{R^2T_o}{4}\left\{-\sin\varphi\left(\frac{1}{D_1}+\frac{1}{D_N}\right)+2\cos\varphi\cdot\varphi/D_N+\sum_{i=1}^{n-1}[2\cos\varphi_i-\right.$$

$$\cdot\varphi_i-\sin(2\varphi_i-\varphi)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)+$$

$$+\frac{R^2M_o}{4}\left\{\cos\varphi\left(\frac{1}{D_1}-\frac{1}{D_n}\right)-\right.$$

$$-2\sin\varphi\cdot\varphi/D_n-\sum_{i=1}^{n-1}\Big[\cos(2\varphi_i-\varphi)+$$

$$\left.+2\sin\varphi\cdot\varphi_i\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\right\}+$$

$$+\frac{R^4q}{4}\left\{\text{cs}\varphi\text{o}\frac{3}{D_1}+\frac{1}{D_n}\right)+$$

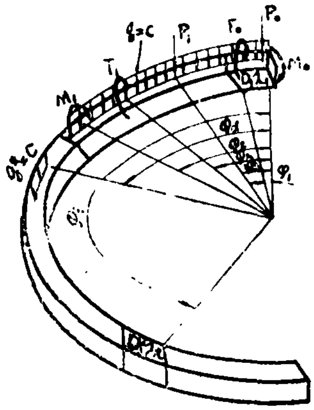

图5

$$+2\sin\varphi\cdot\varphi-4)/D_n+\sum_{i=1}^{n-1}\Big[-4\cos(\varphi-\varphi_i)+\cos(2\varphi_i-\varphi)+2\sin\varphi\cdot\varphi_i\Big]$$

$$\cdot\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+\frac{R^3P_o}{4}\left\{-3\sin\varphi\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_n}\right)+(4\varphi+2\cos\varphi\cdot\varphi)\right.$$

$$\cdot\frac{1}{\lambda_n}+\sum_{i=1}^{n-1}\Big[4\varphi_i-4\sin\varphi_i-4\sin(\varphi_i-\varphi)+2\cos\varphi\cdot\varphi_i+$$

$$+\sin(2\varphi_i-\varphi)\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)+$$

$$+\frac{R^2T_o}{4}\left\{\sin\varphi\left(\frac{1}{\lambda_1}+\frac{-3}{\lambda_n}\right)+2\cos\varphi\cdot\varphi/\lambda_n+\sum_{i=1}^{-1}[-4\sin\varphi_i+\right.$$

$$+2\cos\varphi.\varphi_i+\sin(2\varphi_i-\varphi)]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^2M_o}{4}\left\{-\cos\varphi\left(\frac{1}{\lambda_1}+\right.\right.$$

$$\left.+\frac{1}{\lambda_n}\right)+(-2\sin\varphi\cdot\varphi+4)/\lambda_n+\sum_{i=1}^{n-1}[-4\cos\varphi_i+\cos(2\varphi_i-\varphi)-$$

$$-2\sin\varphi\cdot\varphi_i]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^4q}{4}\left\{\cos\varphi\frac{5}{\lambda_1}+\frac{3}{\lambda_n}\right)-4\left(\frac{1}{\lambda_1}+\right.$$

$$\left.+\frac{1}{\lambda_n}\right)+2\varphi(\varphi+\sin\varphi)/\lambda_n+\sum_{i=1}^{n-1}[2\varphi_i^2-4\varphi_i\sin(\varphi_i-\varphi)-4\cos(\varphi_i-\varphi)+$$

$$+4\cos\varphi_i-\cos(2\varphi_i-\varphi)+2\sin\varphi\cdot\varphi_i]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+H(\varphi-$$

$$-\varphi_a)\frac{R^3P_1}{4}\{[\sin(\varphi-\varphi_a)-2\cos(\varphi-\varphi_a)\cdot\varphi_a]/D_a+[2\cos(\varphi-\varphi_a)\cdot\varphi-$$

$$-\sin(\varphi-\varphi_a)]/D_n+\sum_{i=a^*}^{n-1}[2\cos(\varphi-\varphi_a)\cdot\varphi_i-\sin(2\varphi_i-\varphi-$$

$$-\varphi_a)]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+\sum_{i=b^*}^{n-1}[2\cos(\varphi-\varphi_b)\cdot\varphi_i-\sin(2\varphi-\varphi$$

$$-\varphi_b)] \left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\} + H(\varphi-\varphi_c)\,\frac{R^2M_1}{4}\,\{[\cos(\varphi-\varphi_c)$$

$$+2\sin(\varphi-\varphi_c)\cdot\varphi_c]/D_c^* + [-\cos(\varphi-\varphi_c)-\sin(\varphi-\varphi_c)\cdot\varphi]/D_n$$

$$+\sum_{i=c^*}^{n-1}[-\cos(2\varphi_i-\varphi-\varphi_c)-2\sin(\varphi-\varphi_c)\cdot\varphi_i]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)+$$

$$+H(\varphi-\varphi_d)\frac{R^4q_*}{4}\{[3\cos(\varphi-\varphi_d)-2\sin(\varphi-\varphi_d)\cdot\varphi_d]/D_d^*+[-4+$$

$$+\cos(\varphi-\varphi_d)+2\sin(\varphi-\varphi_d)\cdot\varphi]/D_n+\sum_{i=d^*}^{n-1}[-4\cos(\varphi-\varphi_i)+\cos(2\varphi_i$$

$$-\varphi-\varphi_d)+2\sin(\varphi-\varphi_d)\cdot\varphi_i]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+$$

$$+H(\varphi-\varphi_a)\frac{R^3P_1}{4}\Big\{\Big(4\varphi-3\sin(\varphi-\varphi_a)+2\cos(\varphi-\varphi_a)\cdot\varphi\Big)/\lambda_n+\Big(-4\varphi_a$$

$$-3\sin(\varphi-\varphi_a)-2\cos(\varphi-\varphi_a)\cdot\varphi_a\Big)/\lambda_a^*+\sum_{i=a^*}^{n-1}\Big(4\varphi_i-4\sin(\varphi_i-\varphi_a)+4\sin$$

$$(\varphi-\varphi_i)+2\cos(\varphi-\varphi_a)\cdot\varphi_i+\sin(2\varphi_i-\varphi-\varphi_a)\Big)\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+H(\varphi-$$

$$-\varphi_b)\,\frac{R^2T_1}{4}\Big\{\Big(-2\cos(\varphi-\varphi_b)\cdot\varphi_b+\sin(\varphi-\varphi_b)\Big)/\lambda_b^*+\Big(-3\sin(\varphi-\varphi_b)+$$

$$+2\cos(\varphi-\varphi_b)\Big)/\lambda_n+\sum_{i=b^*}^{n-1}\Big(-4\sin(\varphi_i-\varphi)+2\cos(\varphi-\varphi_b)\cdot\varphi_i+\sin(2\varphi_i-$$

$$-\varphi-\varphi_b)\Big)\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+H(\varphi-\varphi_c)\frac{R^2M_1}{4}\Big\{\Big(4-\cos(\varphi-\varphi_c)+$$

$$+2\sin(\varphi-\varphi_c)\cdot\varphi_c\Big)/\lambda_c^*+\Big(-3\cos(\varphi-\varphi_c)-2\sin(\varphi-\varphi_c)\Big)/\lambda_n+\sum_{i=c^*}^{n-1}$$

$$\Big\{\Big(-4\cos(\varphi_i-\varphi_c)+\cos(2\varphi_i-\varphi-\varphi_c)-2\sin(\varphi-\varphi_c)\cdot\varphi_i\Big)\Big(\frac{1}{\lambda_i}-$$

$$\frac{1}{\lambda_{i+1}}\Big)\Big\}+H(\varphi-\varphi_d)\frac{R^4q^*}{4}\{[-4+5\cos(\varphi-\varphi_d)-2\sin(\varphi-\varphi_d)\cdot\varphi]/\lambda_d^*+$$

$$+[-4+3\cos(\varphi-\varphi_d)+2(\varphi-\varphi_d)^2+2\sin(\varphi-\varphi_d)\cdot\varphi]/\lambda_n+\sum_{i=d^*}^{n-1}[2(\varphi_i$$

$$-\varphi_d)^2-4(\varphi_i-\varphi_d)\sin(\varphi-\varphi_i)-4\cos(\varphi-\varphi_i)+4\cos(\varphi_i-\varphi_d)-\cos(2\varphi_i$$

$$-\varphi-\varphi_d)+2\sin(\varphi-\varphi_d)\cdot\varphi_i]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\} \qquad (3-1)$$

$$R\alpha(\varphi)=R\alpha_0\cos\varphi+R\beta_0\sin\varphi+$$

$$+\frac{R^3P_0}{4}\Big(-\cos\varphi\left(\frac{1}{D_1}-\frac{1}{D_n}\right)-2\sin\varphi.\varphi/D_n+\sum_{i=1}^{n-1}\Big(\cos(2\varphi i-\varphi)-2\sin\varphi.\varphi$$

$$i\Big]\left(\frac{1}{Di}-\frac{1}{D_{i+1}}\right)\Bigg\}+\frac{R^2T_o}{4}\Bigg\{-\cos\varphi\left(\frac{1}{D_1}\frac{1}{D_n}\right)-2Sin\varphi.\varphi/D_n+\sum_{i=1}^{n-1}\Big[co(\ _i$$

$$-\varphi)-2Sin\varphi.\varphi_i\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Bigg\}-\frac{R^2M_o}{4}\Big\{Sin\varphi\left(\frac{1}{D_1}+\frac{1}{D_n}\right)+2cos\varphi.\varphi/D_n+$$

$$+\sum_{i=1}^{n-1}\Big[2cos\varphi.\varphi_i+Sin(2\varphi_i-\varphi)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-\frac{R^4q}{4}\Big\{Sin\varphi\left(\frac{3}{D_1}-\frac{1}{D_n}\right)$$

$$-2cos\varphi.\varphi/D_n+\sum_{i=1}^{n-1}\Big[4Sin(\varphi_i-\varphi)-2cos\varphi.\varphi_i-Sin\ (2\varphi_i-\varphi)\Big]\left(\frac{1}{D_i}-\right.$$

$$\left.-\frac{1}{D_{i+1}}\right)\Big\}+\frac{R^3P_o}{4}\Big\{-3cos\varphi/\lambda+(4-cos4-2sin\varphi.\varphi)/\lambda_n+\sum_{i=1}^{n-1}\Big[4cos(\varphi_i$$

$$-\varphi)+cos\ (2\varphi_i-\varphi)-2sin\varphi.\varphi_i\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^2T_o}{4}\Big\{Cos\varphi/\lambda_1+$$

$$+(Cos\varphi-2sin\varphi.\varphi)/\lambda n-\sum_{i=1}^{n-1}[cos(2\varphi_i-\varphi)+2sin\varphi.\varphi_i]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^2M_o}{4}$$

$$\Big\{sin\varphi/\lambda_1-(2cos\varphi.\varphi-sin\varphi)/\lambda_n-\sum_{i=1}^{n-1}[2cos\varphi.\varphi_i-sin(2\varphi_i\varphi-\varphi)]$$

$$\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^4q}{4}\Big\{-5sin\varphi/\lambda_1+(4\varphi-2cos\varphi.\varphi-sin\varphi)/\lambda_n+$$

$$+\sum_{i=1}^{n-1}\Big[4\varphi_i cos\ (\varphi_i-\varphi)-4sin(\varphi_i-\varphi)+2cos\varphi.\varphi_i-sin\ (2\varphi_i-\varphi)\Big]$$

$$\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+H(\varphi-\varphi_a)\frac{R^3P_1}{4}\Big\{\Big[-cos\ (\varphi-\varphi_a)+2sin\ (\varphi-\varphi_a).$$

$$\varphi_a]/Da\cdot+\Big[cos\ (\varphi-\varphi_a)-2sin\ (\varphi-\varphi_a)-\varphi/Dn+\sum_{i=a^*}^{n-1}\Big[cos\ (2\varphi_i-$$

$$\varphi-\varphi_a)-2sin\ (\varphi-\varphi_a)\ .\varphi i\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+$$

$$+H\ (\varphi-\varphi_b)\ \frac{R^2T_1}{4}\Big\{\Big[-cos\ (\varphi-\varphi_b)+2sin\ (\varphi-\varphi_b).$$

$$\varphi_b\Big]/D_b\cdot+\Big[cos(\varphi-\varphi_b)-2sin\ (\varphi-\varphi_b)\ .\ \varphi\Big]/D_n+\sum_{i=\ \cdot}^{n-1}\Big[cos\ (2\varphi_i-\varphi$$

$$-\varphi_b)-2sin\ (\varphi-\varphi_b)\ .\varphi_i\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-$$

$$-H\ (\varphi-\varphi_c)\ \frac{R^2M_1}{4}\Big\{\Big[-2cos(\varphi-\varphi_c)\ .\varphi_c+sin(\varphi-$$

$$\varphi_c)\Big]/D_c^* + \Big[2cos(\varphi-\varphi_c)\cdot\varphi + sin(2\varphi-\varphi_c)\Big]/D_n +$$

$$\sum_{i=c^*}^{n-1}[2cos(\varphi-\varphi_c)\cdot\varphi_i + sin(2\varphi_i-\varphi-\varphi_c)]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\} -$$

$$-H(\varphi-\varphi_d)\frac{R^4q^*}{4}\Big\{\Big[-3sin(\varphi-\varphi_d) - 2cos(\varphi-\varphi_d)\cdot\varphi_d\Big]/D_d^* +$$

$$+\Big[2cos(\varphi-\varphi_d)\cdot\varphi + sin(\varphi-\varphi_d)\Big]/D_n + \sum_{i=d^*}^{n-1}\Big[4sin(\varphi-\varphi_i) + 2cos(\varphi-$$

$$-\varphi_d)\cdot\varphi_i + sin(2\varphi_i-\varphi-\varphi_d)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\} + H(\varphi-\varphi_a)\frac{R^3P_1}{4}$$

$$\Big\{\Big[-3cos(\varphi-\varphi_a) + 2sin(\varphi-\varphi_a)\cdot\varphi_a\Big]/\lambda_a^* + \Big[4-cos(\varphi-\varphi_a) - 2sin(\varphi-$$

$$\varphi_a)\cdot\varphi\Big]/\lambda_n + \sum_{i=a^*}^{n-1}\Big[4cos(\varphi_i-\varphi) - cos(2\varphi_i-\varphi-\varphi_a) - 2sin(\varphi-\varphi_a)\cdot\varphi_i\Big]$$

$$\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\} + H(\varphi-\varphi_b)\frac{R^2T_1}{4}\Big\{\Big[cos(\varphi-\varphi_b) + 2sin(\varphi-\varphi_b)\cdot\varphi_b.\Big]$$

$$/\lambda_a^* + \Big[-cos(\varphi-\varphi_b) - 2sin(\varphi-\varphi_b)\cdot\varphi\Big]/\lambda_n + \sum_{i=b^*}^{n-1}\Big[-cos(2\varphi_i-\varphi-\varphi_b) -$$

$$-2sin(\varphi-\varphi_b)\cdot\varphi_i\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\} + H(\varphi-\varphi_c)\frac{R^2M_1}{4}\Big\{\Big[2cos(\varphi-$$

$$\varphi_c)\cdot\varphi_c + sin(\varphi-\varphi_c)\Big]/\lambda_c^* - \Big[2cos(\varphi-\varphi_c)\cdot\varphi - sin(\varphi-\varphi_c)\Big]/\lambda_q -$$

$$\sum_{i=c^*}^{n-1}\Big[2cos(\varphi-\varphi_c)\cdot\varphi_i + sin(2\varphi_i-\varphi-\varphi_c)\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\} + H(\varphi-$$

$$\varphi_d)\frac{R^4q^*}{4}\Big\{\Big[-5sin(\varphi-\varphi_d) - 2cos(\varphi-\varphi_d)\cdot\varphi_d\Big]/\lambda^*_d + \Big[4(\varphi-\varphi_d) +$$

$$+2cos(\varphi-\varphi_d)\cdot\varphi - sin(\varphi-\varphi_d)\Big]/\lambda_n + \sum_{i=d^*}^{n-1}\Big[4(\varphi_i-\varphi_d)cos(\varphi-\varphi_i) + 4sin$$

$$(\varphi-\varphi_i) + 2cos(\varphi-\varphi_d)\cdot\varphi_i - sin(2\varphi_i-\varphi-\varphi_d)\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\} \tag{3-I}$$

$$R\beta(\varphi) = -R\alpha_0 sin\varphi + R\beta_0 cos\varphi - \frac{R^3P_0}{4}\Big\{-sin\varphi\left(\frac{1}{D_1}+\frac{1}{D_n}\right) + 2cos\varphi.\varphi$$

$$/D_n + \sum_{i=1}^{n-1}\Big[2cos\varphi.\varphi_i - sin(2\varphi_i-\varphi)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}$$

$$-\frac{R^2T_0}{4}\Big\{-Sin\left(\frac{1}{D_1}+\frac{1}{D_n}\right) + 2cos\varphi\cdot\varphi/D_n +$$

$$+\sum_{i=1}^{n-1}\left[2\cos\varphi\cdot\varphi_i-\sin(2\varphi_i-\varphi)\right]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-\frac{R^2M_o}{4}\left\{\cos\varphi\frac{1}{D_i}+\frac{1}{D_n}\right)-$$

$$-2\sin\varphi\cdot\varphi/D_n+\sum_{i=1}^{n-1}\left[-\cos(2\varphi_i-\varphi)-2\sin\varphi\cdot\varphi_i\right]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-$$

$$-\frac{R^4q}{4}\left\{\cos\varphi\left(\frac{3}{D_1}+\frac{1}{D_n}\right)+2\sin\varphi\cdot\varphi/D_n+{}^{-4}/D_1+\sum_{i=1}^{n-1}\Big[\cos(2\varphi_i-\varphi)+\right.$$

$$+2\sin\varphi\cdot\varphi_i\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+\frac{R^3P_o}{4}\Big\{3\sin\varphi/\lambda_1+(-2\cos\varphi\cdot\varphi)/\lambda_n$$

$$+\sum_{i=1}^{n-1}\Big[-2\cos\varphi\cdot\varphi_i-\sin(2\varphi_i-\varphi)+4\sin(\varphi_i-\varphi)\cdot\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+$$

$$+\frac{R^2M_o}{4}\Big\{\cos\varphi/\lambda_1+(-\cos\varphi+2\sin\varphi\cdot\varphi)/\lambda_n+\sum_{i=1}^{n-1}\Big[-\cos(2\varphi_i-\varphi)+$$

$$+2\sin\varphi\cdot\varphi_i\Big]\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}+\frac{R^4q}{4}\Big\{-5\cos\varphi/\lambda_1+(4+\cos\varphi-2\sin\varphi\cdot\varphi)/\lambda_n+$$

$$+\sum_{i=1}^{n-1}\Big[\varphi_i\sin(\varphi_i-\varphi)+4\cos(\varphi_i-\varphi)+\cos(2\varphi_i-\varphi)-2\sin\varphi\cdot\varphi_i\Big]$$

$$\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)\Big\}-H(\varphi-\varphi_a)\frac{R^3P_1}{4}\Big\{-\Big[2\cos(\varphi-\varphi_a)\cdot\varphi_a+$$

$$+\sin(\varphi-\varphi_a)\Big]/D_{a^*}+\left[2\cos(\varphi-\varphi_a)\cdot\varphi-\sin(\varphi-\varphi_a)\right]/D_n+\sum_{i=a^*}^{n-1}$$

$$\Big[2\cos(\varphi-\varphi_a)\cdot\varphi_i-\sin(2\varphi_i-\varphi-\varphi_a)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-$$

$$-H(\varphi-\varphi_b)\frac{R^2T_1}{4}\Big\{-\left[2\cos(\varphi-\varphi_b)\cdot\varphi_b+\sin(\varphi-\varphi_b)\right]/D_{b^*}+$$

$$\left[2\cos(\varphi-\varphi_b)\cdot\varphi-\sin(\varphi-\varphi_b)\right]/D_n+\sum_{i=b^*}^{n-1}\Big[2\cos(\varphi-\varphi_b)\cdot\varphi_i$$

$$-\sin(2\varphi_i-\varphi-\varphi_b)\Big]\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-H(\varphi-\varphi_c)\frac{R^2M_1}{4}$$

$$\Big\{\left[\cos(\varphi-\varphi_c)+2\sin(\varphi-\varphi_c)\cdot\varphi_c\right]/D_{c^*}-\Big[\cos(\varphi-\varphi_c)+$$

$$+2\sin(\varphi-\varphi_c)\cdot\varphi/D_n+\sum_{i=c^*}^{n-1}\Big[-\cos(2\varphi_i-\varphi-\varphi_c)-2\sin(\varphi-\varphi_c)\cdot\varphi_i\Big]$$

$$\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}-H(\varphi-\varphi_d)\frac{R^4q^*}{4}\Big\{\Big[3\cos(\varphi-\varphi_d)-2\sin(\varphi-\varphi_d)\cdot$$

$$\cdot\varphi_d\Big]/D_{d^*}+\Big[-4+\cos(\varphi-\varphi_d)+2\sin(\varphi-\varphi_d)\cdot\varphi\Big]/D_n+\sum_{i=d^*}^{n-1}\Big[-4\cos(\varphi-$$

$$\varphi_i)+\cos(2\varphi_i-\varphi-\varphi_d)+2\sin(\varphi-\varphi_d)\cdot\varphi_i\Big]\cdot\left(\frac{1}{D_i}-\frac{1}{D_{i+1}}\right)\Big\}+H(\varphi-$$

$\varphi_a)\frac{R^3P_1}{4}\Big\{\Big[2\cos(\varphi-\varphi_a)\cdot\varphi_a+3\sin(\varphi-\varphi_a)\Big]/\lambda_{a^*}+\Big[2\cos(\varphi-\varphi_a)\cdot\varphi+$

$\sin(\varphi-\varphi_a)\Big]/\lambda_n+\sum_{i=1}^{n-1}\Big[-2\cos(\varphi-\varphi_a)\cdot\varphi_i-\sin(2\varphi_i-\varphi-\varphi_a)+4\sin(\varphi_i-$

$\varphi)\Big]\Big(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\Big)\Big\}-H(\varphi-\varphi_a)\frac{R^2T_1}{4}\Big\{\Big[2\cos(\varphi-\varphi_b)\cdot\varphi_b+\sin$

$(\varphi-\varphi_b)\Big]/\lambda_{b^*}+\Big[2\cos(\varphi-\varphi_b)\cdot\varphi+\sin(\varphi-\varphi_b)\Big]/\lambda_n+\sum_{i=b}^{n-1}\cdot\Big[2\cos$

$(\varphi-\varphi_b)\cdot\varphi_i+\sin(2\varphi_i-\varphi-\varphi_b)\Big(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\Big)\Big\}+H(\varphi-\varphi_c)$

$\frac{R^2M_1}{4}\Big\{\Big[\cos(\varphi-\varphi_c)-2\sin(\varphi-\varphi_c)\cdot\varphi_c\Big]/\lambda_{c^*}+\Big[-\cos(\varphi-\varphi_c)$

$+2\sin(\varphi-\varphi_c)\cdot\varphi\Big]/\lambda_n+\sum_{i=c^*}^{n-1}\Big[-\cos(2\varphi_i-\varphi-\varphi_c)+2\sin(\varphi-\varphi_c)\cdot$

$\varphi_i\Big]\Big(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\Big)\Big\}+H(\varphi-\varphi_d)\frac{R^4q^*}{4}\Big\{\Big[-5\cos(\varphi-\varphi_d)+2\sin$

$(\varphi-\varphi_d)\cdot\varphi_d\Big]/\lambda_{d^*}+\Big[4-2\sin(\varphi-\varphi_d)\cdot\varphi+\cos(\varphi-\varphi_d)\Big]/\lambda_n+\sum_{i=d}^{n-1}$

$\Big[4(\varphi_i-\varphi_d)\cdot\sin(\varphi_i-\varphi)+4\cos(\varphi_i-\varphi)-2\sin(\varphi-\varphi_d)\cdot\varphi_i+\cos(2\varphi_i-$

$\varphi-\varphi_d)\Big]\Big(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\Big)\Big\}$ (3—Ⅱ)

式中：

φ_i——初始点到第i段惯性矩终点的角度，

D_i、λ_i、φ_i——均为常数（$i=1$，2…

a^*—d^*是一种记号。例如弯矩作用在第2段惯性矩上。如图6所示，则$C^*=2$。

φ——全区域变化的量，

n——分段变化的量，（在例题中说明）。

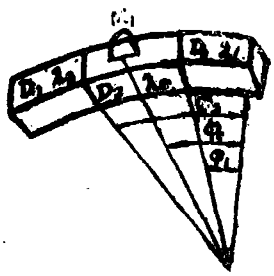

图 6

以上我们得到了公式（3—1）—（3—Ⅱ），利用这组公式即可解决载荷垂直于静定和静不定阶梯园环的各种计算。

四、等截面弹性园环弯扭的一般解

令$D_1 = D_n = D$，由（3—I）式得

$$U_z(\varphi) = U_{z0} + R\alpha_0\sin\varphi + R\beta_0(1-\cos\varphi) + \frac{R^2}{2D}\{RP_0[\cos\varphi\cdot\varphi - \sin\varphi] +$$
$$+ T_0(\cos\varphi\cdot\varphi - \sin\varphi) - M_0\sin\varphi\cdot\varphi + R^2q(-2+\sin\varphi\cdot\varphi + 2\cos\varphi)\} +$$
$$\frac{R^2}{2\lambda}\{RP_0(2\varphi + \cos\varphi\cdot\varphi - 3\sin\varphi) + T_0(\cos\varphi\cdot\varphi - \sin\varphi) + M_0(2-\sin\varphi\cdot\varphi - 2\cos\varphi)$$
$$+ R^2q(-4+4\cos\varphi + \varphi^2 + \sin\varphi\cdot\varphi)\} + \frac{R^2}{2D}\{H(\varphi-\varphi_a)RP_1[-\sin(\varphi-\varphi_a) +$$
$$+\cos(\varphi-\varphi_a)\cdot(\varphi-\varphi_a)] + H(\varphi-\varphi_b)T_1[-\sin(\varphi-\varphi_b)\varphi) + \cos(\varphi-\varphi_b)\cdot(\varphi-\varphi_b)] +$$
$$+ H(\varphi-\varphi_c)M_1\cdot[-\sin(\varphi-\varphi_c)\cdot(\varphi-\varphi_c)] + H(\varphi-\varphi_d)R^2q[-2+\sin(\varphi-\varphi_d)\cdot$$
$$(\varphi-\varphi_d) + 2\cos(\varphi-\varphi_d)]\} + \frac{R^2}{2\lambda}\Big\{H(\varphi-\varphi_a)RP_1[2(\varphi-\varphi_a) + \cos(\varphi-\varphi_a)\cdot$$
$$\cdot(\varphi-\varphi_a) - 3\sin(\varphi-\varphi_a)] + H(\varphi-\varphi_b)T_1[\cos(\varphi-\varphi_b)\cdot(\varphi-\varphi_b) - \sin(\varphi-\varphi_b)] +$$
$$+ H(\varphi-\varphi_c)M_1[2-\sin(\varphi-\varphi_c)\cdot(\varphi-\varphi_c) - 2\cos(\varphi-\varphi_c)] + H(\varphi-\varphi_d)R^2q[-4+$$
$$+4\cos(\varphi-\varphi_d) + (\varphi-\varphi_d)^2 + \sin(\varphi-\varphi_d)\cdot(\varphi-\varphi_b)]\} \qquad (4\text{—I})$$

由（3—I）式得

$$R\alpha(\varphi) = R\alpha_0\cos\varphi + R\beta_0\sin\varphi - \frac{R^2}{2D}\{RP_0\sin\varphi\cdot\varphi + T_0\sin\varphi\cdot\varphi + M_0(\sin\varphi$$
$$+\cos\varphi\cdot\varphi) + R^2q(\sin\varphi - \cos\varphi\cdot\varphi)\} + \frac{R^2}{2\lambda}\{RP_0(2-2\cos\varphi - \sin\varphi\cdot\varphi) +$$
$$+ T_0(-\sin\varphi\cdot\varphi) + M_0(\sin\varphi - \cos\varphi\cdot\varphi) + R^2q(2\varphi + \cos\varphi - 3\sin\varphi)\} -$$
$$-\frac{R^2}{2D}\{H(\varphi-\varphi_a)RP_1\sin(\varphi-\varphi_a)\cdot(\varphi-\varphi_a) + H(\varphi-\varphi_b)T_1\sin(\varphi-\varphi_b)\cdot(\varphi-\varphi_b) +$$
$$+ H(\varphi-\varphi_c)M_1[\cos(\varphi-\varphi_c)\cdot(\varphi-\varphi_c) + \sin(\varphi-\varphi_c)] + H(\varphi-\varphi_d)$$
$$R\ q^*[\sin(\varphi-\varphi_d) - \cos(\varphi-\varphi_d)\cdot(\varphi-\varphi_d)]\} + \frac{R^2}{2\lambda}\{H(\varphi-\varphi_a)$$
$$RP_1[2-2\cos(\varphi-\varphi_a) - \sin(\varphi-\varphi_a)\cdot(\varphi-\varphi_a)] + H(\varphi-\varphi_b)T_1[-\sin(\varphi-\varphi_b)$$
$$\cdot(\varphi-\varphi_b)] + H(\varphi-\varphi_c)M_1[\sin(\varphi-\varphi_c) - \cos(\varphi-\varphi_c)\cdot(\varphi-\varphi_c)] + H(\varphi-\varphi_d)$$
$$R^2q^*[2(\varphi-\varphi_d) + \cos(\varphi-\varphi_d) - 3\sin(\varphi-\varphi_d)]\} \qquad (4\text{—I})$$

$$R\beta(\varphi) = R\beta_0\cos\varphi - R\alpha_0\sin\varphi + \frac{R^2}{2D}\{RP_0(\sin\varphi - \cos\varphi\cdot\varphi) + T_0(\sin\varphi - \cos\varphi\cdot\varphi)$$
$$+ M_0\sin\varphi\cdot\varphi + R^2q(2-\sin\varphi\cdot\varphi - 2\cos\varphi)\} + \frac{R^2}{2\lambda}\{RP_0(-\cos\varphi\cdot\varphi + \sin\varphi)$$
$$- T_0(\cos\varphi\cdot\varphi + \sin\varphi) + M_0\sin\varphi\cdot\varphi + R^2q(2-2\cos\varphi - \cos\varphi\cdot\varphi)] + \frac{R^2}{2D}\cdot$$
$$\cdot\{H(\varphi-\varphi_a)RP_1[\sin(\varphi-\varphi_a) - \cos(\varphi-\varphi_a)\cdot(\varphi-\varphi_a)] + H(\varphi-\varphi_b)$$
$$T_1[\sin(\varphi-\varphi_d) - \cos(\varphi-\varphi_d)\cdot(\varphi-\varphi_d)] + H(\varphi-\varphi_c)M_1[\sin(\varphi-\varphi_c\cdot(\varphi-\varphi_c)$$

$$+R^2q^*[2-2cos(\varphi-\varphi_d)-sin(\varphi-\varphi_d)\cdot(\varphi-\varphi_d)]\}+\frac{R^2}{2\lambda}\{H(\varphi-\varphi_a)$$

$$RP_1[-cos(\varphi-\varphi_a)\cdot(\varphi-\varphi_a)+sin(\varphi-\varphi_a)]-H(\varphi-\varphi_b)T_1[cos(\varphi-\varphi_b)$$
$$\cdot(\varphi-\varphi_b)+sin(\varphi-\varphi_b)]+H(\varphi-\varphi_c)M_1 sin(\varphi-\varphi_c)\cdot(\varphi-\varphi_c)+$$
$$+H(\varphi-\varphi_d)R^2q^*[2-2cos(\varphi-\varphi_d)-sin(\varphi-\varphi_d)\cdot(\varphi-\varphi_d)]\}\qquad(4—Ⅱ)$$

以上我们得到了（4—Ⅰ）～（4—Ⅱ），利用这组公式即可解决载荷垂直于静定或静不定等截面弹性园环的各种计算。

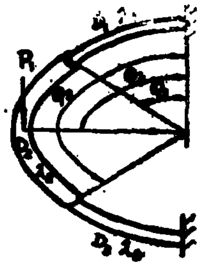

图7

五、简　　例

如图7所示，已知$U_{z0}=R\alpha_0=R\beta_0=0$　$U_z(\pi)=R\alpha(\pi)=R\beta(\pi)=0$　$D_1=D$、$D_2=2D$、$\lambda_1=\lambda$、$\lambda_2=2\lambda$、$D_3=D$、$\lambda_3=\lambda$、$\varphi_1=\frac{\pi}{4}$、$\varphi_2=\frac{3\pi}{4}$、$\varphi_a=\frac{\pi}{2}$、$a^x=2$、为简便起见，设$\frac{D}{\lambda}=1$。

试求P_0、T_0、M_0及挠度、倾角的一般方程。

〔解〕：　$\varphi=\pi$时，取$n=3$代入公式（3—Ⅰ）～（3—Ⅱ）得：

$$\left.\begin{aligned}&-3\pi T_0+(8-4\sqrt{2})M_0=(\frac{3\pi}{2}-6)RP_1\\&3\pi RP_0+3\pi T_0=(2+\sqrt{2})RP_1\\&(8-4\sqrt{2})RP_0+3\pi M_0=(-\frac{3\pi}{2}+4-\sqrt{2})RP_1\end{aligned}\right\}\qquad(5—1)$$

由（A—4）式得：

$$T(\varphi)=\frac{-27\pi^3+18\pi^2\sqrt{2}+48\pi(5-3\sqrt{2})}{2[27\pi^3-96\pi(3-2\sqrt{2})]}RP_1 sin\varphi+$$

$$+\frac{-36\pi^2(\frac{\sqrt{2}}{2}+1)+64(2-\sqrt{2})}{2[27\pi^3-96\pi(3-2\sqrt{2})]}RP_1 cos\varphi+$$

$$+\frac{27\pi^2-18\pi^2\sqrt{2}+48\pi(5-3\sqrt{2})}{2[27\pi^3-96\pi(3-2\sqrt{2})]}RP_1+H(\varphi-\frac{\pi}{2})RP_1[1-cos(\varphi-\frac{\pi}{2})]$$

$$M(\varphi)=\frac{-27\pi^3+18\pi^2\sqrt{2}+48\pi(5-3\sqrt{2})}{27\pi^3-96\pi(3-2\sqrt{2})}RP_1 cos\varphi$$

$$+\frac{36\pi^2(\frac{\sqrt{2}}{2}+1)-64(2-\sqrt{2})}{2[27\pi^3-96\pi(3-2\sqrt{2})]}RP_1 sin\varphi+H(\varphi-\frac{\pi}{2})RP_1 sin(\varphi-\frac{\pi}{2})$$

由（3－Ⅰ）式得：

$$U_z(\varphi)=\frac{36\pi^2(\frac{\sqrt{2}}{2}+1)-64(2-\sqrt{2})}{8[27\pi^3-96\pi(3-2\sqrt{2})}R^3P_1\{-\sin\varphi(\frac{1}{D_1}+\frac{1}{D_n}+2\cos\varphi\cdot\varphi/D_n$$

$$+\sum_{j=1}^{n-1}[2\cos\varphi\cdot\varphi_i-\sin(2\varphi_i-\varphi)](\frac{1}{D_i}-\frac{1}{D_{i+1}})\}$$

$$+\frac{-27\pi^3+18\pi^2\sqrt{2}+48\pi(5-3\sqrt{2})}{8(27\pi^3-96\pi(3-2\sqrt{2}))]}R^3P_1\{\cos\varphi(\frac{1}{D_1}-\frac{1}{D_n})-2\sin\varphi\cdot\varphi/D_n\}$$

$$-\sum_{j=1}^{n-1}[\cos(2\varphi_i-\varphi)+2\sin\varphi\cdot\varphi_i]\quad(\frac{1}{D_i}-\frac{1}{D_{i+1}})\}+$$

$$+\frac{36\pi^2\ (\frac{\sqrt{2}}{2}+1)-64(2-\sqrt{2})}{8(27\pi^3-96\pi(3-2\sqrt{2}))]}R^3P_1\{\sin\varphi(\frac{1}{\lambda_1}+\frac{-3}{\lambda_n}+2\cos\varphi\cdot\varphi/\lambda_n+$$

$$+\sum_{j=1}^{n-1}[-4\sin\varphi_i+2\cos\varphi\cdot\varphi_i+\sin(2\varphi_i-\varphi)](\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}})\}+$$

$$+\frac{27\pi^3-18\pi^2\sqrt{2}+48\pi\ (5-3\sqrt{2})}{8[27\pi^3-96\pi(3-2\sqrt{2})]}RP_1\{-4\sin\varphi/\lambda_1+4\varphi/\lambda_n$$

$$+\sum_{i=1}^{n-1}[4\varphi_i-4\sin(\varphi_i-\varphi)\cdot\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}}\right)+$$

$$+\frac{-27\pi^3+18\pi^2\sqrt{2}+48\pi(5-3\sqrt{2})}{8(27\pi^3-96\pi(3-2\sqrt{2}))]}R^3P_1\{-\cos\varphi(\frac{1}{\lambda_1}+\frac{1}{\lambda_n})$$

$$-2\sin\varphi\cdot\varphi/\lambda_n+4/\lambda_1+\sum_{i=1}^{n-1}\quad[-4\cos\varphi_i+\cos(2\varphi_i-\varphi)$$

$$-2\sin\varphi\cdot\varphi_i](\frac{1}{\lambda_1}-\frac{1}{\lambda_{i+1}})\}+H(\varphi-\frac{\pi}{2})\frac{R^3P_1}{4}\{[\sin(\varphi-\frac{\pi}{2})-2\cos(\varphi-\frac{\pi}{2})$$

$$\cdot\frac{\pi}{2}]/D_2+[2\cos(\varphi-\frac{\pi}{2})\cdot\varphi-\sin(\varphi-\frac{\pi}{2})]/D_n+\sum_{i=2}^{n-1}[2\cos(\varphi-\frac{\pi}{2})\cdot\varphi_i-$$

$$-\sin(2\varphi_i-\varphi-\frac{\pi}{2})]\ (\frac{1}{D_i}-\frac{1}{D_{i+1}})\ \}+H(\varphi-\frac{\pi}{2})\frac{R^3P_1}{4}\{[-2\pi-3\sin(\varphi-\frac{\pi}{2})$$

$$-2\cos(\varphi-\frac{\pi}{2})\cdot\frac{\pi}{2}]/\lambda_2+[4\varphi-3\sin(\varphi-\frac{\pi}{2})+2\cos(\varphi-\frac{\pi}{2})\cdot\varphi]/\lambda_n+$$

$$+\sum_{i=2}^{n-1}[4\varphi_i-4\sin(\varphi_i-\frac{\pi}{2})+4\sin\ (\varphi-\varphi_i)+2\cos(\varphi-\frac{\pi}{2})\ \cdot\varphi_i+$$

$$+\sin(2\varphi_i-\varphi-\frac{\pi}{2})]\ (\frac{1}{\lambda_i}-\frac{1}{\lambda_{i+1}})\ \}$$

用P_0、T_0、M_0代入（3—Ⅰ）、（3—Ⅱ）式就可得$R\alpha(\varphi)$、$R\beta(\varphi)$的一般方程式，因篇幅关系，在此略。

求得了$U_z(\varphi)$、$R\alpha(\varphi)$、$R\beta(\varphi)$方程后，只要改变n及φ就可得到不同截面的挠度和倾角。

以上一些结果与卡氏定理求得的结果完全一致。

注意本文示例。 $0<\varphi\leqslant\frac{\pi}{4}$， $n=1$。

$\frac{\pi}{4}<\varphi\leqslant\frac{3\pi}{4}$， $n=2$。 $\frac{3\pi}{4}<\varphi\leqslant\pi$， $n=3$。

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Bending and Torsion problems of Non-homogeous Elastic Circular Ring with Variable Cross-Section under Vertical Loods

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Abstract

Based on the fundamental differential equations, and with the application of the differential relationship the internal force and displacement, this paper investigates all-roundly the bending and torsion problems of non-homogeneous elastic circular ring with variable cross-section under vertical loods. The general integral formulae are obtained for the bending and torsional stiffness expressed in the formulae arbitrary functions, through appropriate mathematical transformations. The general solutions for two special case i.e the cicrular ring with stepped and with constand cross-section, are derived. Lastly, for illustration of the concrete application of these formulae, an expositive example is given and soloved.